Consequences of a Discrete Space-Time
(MTF071)

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June 26, 2000

Abstract

We will look for diverging integrals in QFT and see if they can be made convergent if the space-time is discrete instead of continous. In fact, what is it that really tells us that space-time should be continous? Everything else is quantized, so why not space-time as well?

Finally we will calculate the propagator for a discretized space and compare it with the propagator of the standard model.

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Chapter 1

Introduction

In quantum field theory there are several non-explained assumptions and experimental facts that has been imposed on the theory without further understanding. The non-symmetry between right and left particles is one example, the renormalization procedure is another. Renormalization is done in order to get rid of infinite quantities in quantum field theory. The idea is simple, to make theory match experiment an infinite quantity is subtracted from the Lagrangian. At first regard, this seems very strange indeed, but a closer look reveals that it's maybe not that strange after all. Nevertheless, it is quite embarassing to be obliged to add an infinite term without any apparent explanation. In this work we propose a way to get rid of the infinities, even though we don't explain the counter terms themselves. Our proposal is to discretize the space-time so that there is a smallest distance between two particles thus giving a maximum momentum-transfer in an interaction. There are several places in the theory where the discretization would be thinkable. In this report we consider the free propagator in a discretized space with a cartesian discretization with equal spacing in all space-time directions. The discretization is done on the action integral.

For reference and as a gleam of the possibilities of discretized space we also include a summary of an article by Maneolito M de Souza regarding discrete gauge fields and how they can be viewed as a prolongation of our normal guage theory. These fields produce effects just as continuous fields, they are perfectly well determined and they have several other nice properties.

The possibility of space-time being discrete is a fascinating prospect and it opens up a whole new field of thoughts and theories. It could even be the third quantization, just as the particles and later the fields were quantized, so would the space-time be. There are other theories to make the infinite terms finite, but few of them are as revolutionary and yet so simple, as the thought of a discrete space-time.

Chapter 2

Quantum Field Theory

2.1 Introduction

This first chapter will be dedicated to explaining the basic features of Quantum Field Theory (QFT). Firstly, we will study the theory of $\lambda\phi^4/4!$ theory, though experimentally not so useful, it provides a simple theoretical framework from which we can understand the problematics of any kind of quantum field theory, like for example quantum electro dynamics. The advantage of the $\lambda\phi^4/4!$ theory is the absence of spin and internal quantum numbers as well as interactions with other particles which easily clouds the principal features of QFT. A basic knowledge of the theory of special relativity is assumed as well as a good comprehension of quantum mechanics.

2.2 Non-interacting $\lambda\phi^4/4!$ theory

The $\lambda\phi^4/4!$ theory is the theory of a massive, spinless, self-interacting scalar particle. The Lagrangian of the theory is given by:

$$L = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \lambda\phi^4/4!, \tag{2.1}$$

where $\phi = \phi(x^4)$ is the wave function of the particle, $x^4 = x, y, z, t$, m is its mass and $\lambda\phi^4/4!$ is the interaction term (the factor 4! is only included to give the derivatives a nice form, obviously this constant term can be compensated for with the interaction constant λ). Further on we will use ϕ, $\phi(x)$ and $\phi(x^4)$ interchangeably. The power 4 in ϕ^4 means that for the particle to interact, it takes three other particles.

Before indulging in the interactions between the particles the non-interacting theory is worth some attention. The Lagrangian of this theory of free particles is

$$L_F = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2, \tag{2.2}$$

where the index F signifies Free. The action to be minimized is generally given by

$$S = \int d^4x L \tag{2.3}$$

(integration over the three space variables and time) where we now have $L = L_F$. When the variational equation

$$\frac{\partial}{\partial x_\mu}\left(\frac{\delta L}{\delta(\partial_\mu \phi(x))}\right) - \frac{\delta L}{\delta \phi(x)} = 0 \tag{2.4}$$

is employed to minimize the action, we find the simplest relativistic equivalent to the Schrödinger equation, namely the so called Klein-Gordon equation:

$$(\Box + m^2)\phi(x) = 0, \tag{2.5}$$

where $\Box = \partial_\mu \partial^\mu$, which is the sum of the second derivatives over time and space. This equation can be solved directly by Fourier transformation. The particle ϕ is to be interpreted as a free particle (because there are no interactions), and it is also called the free propagator.

In summary, we have taken the Lagangian, which is the kinetic energy and the potential energy, applied the variational equations to minimize the total energy and thus found an equation describing the free particle, ϕ.

2.3 Interacting $\lambda\phi^4/4!$ theory

Let us now attack the self-interacting theory with the Lagrangian given by 2.1. With the same procedure as for the free particle we get the variational equation:

$$(\Box + m^2)\phi(x) = -\lambda\phi^3/3!, \tag{2.6}$$

which is a non-linear partial differential equation. There is no general way to solve this equation and to calculate the field ϕ we will have to rely on an approximation. In order to linearize the problem we employ perturbation theory.

2.4 Perturbation Theory

2.4.1 Introduction

What we are interested in here is not primarily to solve the non-linear partial differential equation, but rather to get some results. Hence, we will want to calculate the probabilities of interactions, the cross-sections, through matrix elements like

$$\langle out\,|T|\,in\rangle = \langle 0|T(\phi(x_1)\phi(x_2)\ldots\phi(x_n))|0\rangle\,, \tag{2.7}$$

where T is the time operator and $|0\rangle$ is the vacuum field, in the case of n interacting particles. The amplitude of the action over a path $x(t)$ is given by

$$exp\left[\frac{i}{\hbar}S(x(t))\right] \tag{2.8}$$

which means that the total amplitude at a given time t is

$$I = \int_A^B exp\left[\frac{i}{\hbar}S(x(t))\right]\Pi dx(t), \tag{2.9}$$

where $\int \Pi dx(t) = \int \cdots \int dx_1 dx_2 \dots dx_n dx_{n+1} \dots$ is the integral over all the possible paths of the particle from point $A = (x_A, y_A, z_A)$ to point $B = (x_B, y_B, z_B)$. In classical mechanics, only the path that minimizes the action would have been important. Now all paths are important, but the path(s) that minimizes the energy is the most important one. This can also be regarded in the light of stationary quantum mechanics. Obviously, all paths must be accounted for, just like all points in space must be accounted for in stationary quantum mechanics $\int_{\Re^3} |\phi|^2 dV$.

In our case, not only the path is important but the field (or particle wave function) in every point in space-time. Hence, we replace the $x(t)$ by $\phi(x) = \phi(x^4)$. With $\hbar = 1$ as usual we get

$$I = \int_A^B exp\left[iS(\phi(x))\right]D[\phi]. \tag{2.10}$$

The term $D[\phi]$ might look frightening, but as we will see later, it will not cause any trouble as it will cancel from the normalization.

2.4.2 Functional Integration

We will now show that with a specific approximation technique[1] (called functional integration) we can get back the free propagator from the free Lagrangian. This indicates that the method is valid and, in fact, it also holds for interacting fields, In other words, it can be used to calculate probabilities for outcomes of interactions. The calculations are done by a Taylor expansion which means that the interacting fields can be expressed in terms of the free propagator.

The reasoning goes as follows. We add a scalar function $J(x)\phi(x)$ to the Lagrangian, thus giving the integral over all possible paths as:

$$F[J] = \int D[\phi]exp\left[i\int d^4x\,(L_F + J\phi)\right]. \tag{2.11}$$

[1] This is an alternative method to the perturbation expansion of Feynman. They both give the same results.

In order to calculate this, we must put the modified Lagrangian

$$L_F + J\phi = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + J\phi \tag{2.12}$$

on a quadratic form[2]. First we concentrate on the term $\partial_\mu\phi\partial^\mu\phi$. Differentiation of $\partial_\mu(\partial_\mu\phi^2)$ gives:

$$\partial_\mu(\partial_\mu\phi^2) = 2(\phi\Box\phi + \partial_\mu\phi\partial^\mu\phi) \tag{2.13}$$

from which we can obtain $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ as

$$\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) = -\frac{1}{2}\phi\Box\phi + \frac{1}{4}\partial_\mu(\partial_\mu\phi^2). \tag{2.14}$$

However, total derivatives, $\partial_\mu(\partial_\mu\phi^2)$, does not contribute to the action, which leaves us with the quadratic form:

$$\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) = -\frac{1}{2}\phi\Box\phi. \tag{2.15}$$

The kinetic term is now $\phi(\Box + m^2)\phi$ and our next step is to incorporate the differential operator $\Box + m^2$ in our field ϕ thus redefining the field, so we would get something like $\phi'\phi'$. The change of variables is

$$\phi(x) = \int dyK(x-y)\phi'(y), \tag{2.16}$$

where $\hat{K}(p) = (p^2 - m^2)^{-1/2}$, $K(x) = \frac{1}{(2\pi)^4}\int dp\hat{K}(p)e^{ipx}$. K is defined in that way because

$$\mathfrak{F}[\Box + m^2] = p^2 - m^2 = \frac{1}{(\hat{K}(p))^2}. \tag{2.17}$$

This means that we have accomplished the appropriate change of variables in momentum space. For a more rigorous derivation, see Nash [1], p. 32f. As for the other terms in the modified Lagrangian, we absorb them in a similar manner, but we refer again the interested reader to Nash [1], p. 33f.

Finally, we arrive at the equation

$$F[J] = exp\left[-\frac{i}{2}\int dxdyJ(x)\Delta_F(x-y)J(y)\right] D\int D[\phi]e^{i\int d^4x\phi^2/2} \tag{2.18}$$

where D is the Fredholm determinant, the equivalent of the Jacobian in $\mathbb{R}^3$ and $\Delta_F(x-y)$ is defined as

$$(\Box + m^)\Delta_F(x-y) = \delta(x-y), \tag{2.19}$$

[2]In matrix-algebra, the quadradic form is $\vec{x}^T A\vec{x}$, where A is a matrix. In this context, a quadradic form is of the form $\phi A\phi$, were A may be an operator. As for the case with the matrices, a linear transformation can achieve $\phi A\phi = \phi'\phi'$ which is our goal.

which is nothing else than the usual scalar particle Green's function. The only difference between the Klein-Gordon equation 2.5 and this equation is that the particle in the Klein-Gordon equation is not time-ordered. We renormalize $F[J]$ as

$$F[J] = \frac{\int D[\phi] exp\left[i \int d^4x \left(L_F + J\phi\right)\right]}{\int D[\phi] exp\left[i \int d^4x L_F\right]} = exp\left[-\frac{i}{2}\int dxdy J(x)\Delta_F(x-y)J(y)\right] \tag{2.20}$$

in order to have $F[0] = 1$.

It is now fairly easy to calculate explicitly that

$$\left.\frac{\delta^2 F[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} = -i\Delta_F(x_1 - x_2) \tag{2.21}$$

which corresponds to a free particle travelling from point x_1 in space-time to point x_2. In analogy with the case of two derivations, $2n$ derivations corresponds to n free particles. We also see that derivation an odd number of times always gives zero. Finally, for the free fields, n free particles is also written as $\langle 0|T(\phi(x_1)\phi(x_2)\ldots\phi(x_n))|0\rangle$, giving us

$$\left.\frac{\delta^n F[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0} = \langle 0|T(\phi(x_1)\ldots\phi(x_n))|0\rangle\,. \tag{2.22}$$

It can be showed that this means that $\left.\frac{\delta^n F[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0}$ can be used as a "basis" for a functional variant of Taylor series expansion. This property shall be used to expand the interacting version of $F[J]$ in terms of free propagators.

Let us now turn to the interactions. We use the same type of functional but replace the free Lagrangian with the total, $L = L_F + L_I$ Lagrangian, thus the interacting version of $F[J]$ becomes

$$Z[J] = \frac{\int D[\phi] exp\left[i \int d^4x \left(L + J\phi\right)\right]}{\int D[\phi] exp\left[i \int d^4x L\right]}. \tag{2.23}$$

Expansion of the numerator and denominator in terms of λ means that we can evaluate, e.g.

$$\left.\frac{(-i)^4\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0} \tag{2.24}$$

which gives the first order terms in the interaction, which can be expanded in terms of

$$\left.\frac{\delta^4 F[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0}$$

and

$$\left.\frac{\delta^8 F[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)\delta J(x_5)\delta J(x_6)\delta J(x_7)\delta J(x_8)}\right|_{J=0}$$

which are two and four particles interacting respectively as we get from equation 2.21. As stated above, this can be proven to give the same result as Feynman perturbation expansion, and the theory has been validated rather thoroughly.

2.4.3 Summary

In this section we have derived a technique for making calculations with a Lagrangian with an interaction term. Without approximation, there is no known way to solve this, but perturbation theory is known to give results in excellent agreement with experiment. The perturbation theory here employed uses functional integration to derive the solutions in terms of the free propagator.

Chapter 3

Renormalization

There are several different sources of infinities in quantum field theory. Some are more severe than others but even the less serious ones are rather uncomfortable. The task of making a sensible theory of the infinities is called renormalization. For some of the renormalizations it can be argued that they should be present even in the absence of infinities, for others that our physics of today may not be appropriate. In any case, the answer to what is really going on is still an issue of uttermost interest.

Here is a list of the known infinties in the quantum field theory for the $\lambda\phi^4/4!$ theory. Even though there is no practical application of the $\lambda\phi^4/4!$ theory, the same principles and types of infinties are encountered in other field theories, like quantum electro dynamics. The infinities will be explained in more detail below.

1. Vacuum-graphs
2. Particle annihilating itself, $\Delta_F(0)$
3. Mass-shift, pole in $m^2 - i\Pi$ instead of in m^2
4. Ultraviolet divergence
5. Infrared divergence

3.1 Vacuum-graphs

These graphs can be illustrated with e.g. the four-particle interaction term, equation 2.24.

$$\left.\frac{(-i)^4\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0} \tag{3.1}$$

which can be calculated to be

$$\frac{\lambda}{4!}\frac{\delta^4 F[0]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\int d^4x \frac{\delta^4 F[0]}{\delta J(x)\delta J(x)\delta J(x)\delta J(x)} + \dots \tag{3.2}$$

In the light of the results from the end of chapter 2 we can see that this represents four particles, and it is (in principle) represented graphically as in figure 3.1

The first part is only two free propagators just as discussed in the end of

Figure 3.1: Interaction of four particles, the graph to the right represents a vacuum graph.

chapter 1. The second part is

$$\int d^4x \frac{\delta^4 F[0]}{\delta J(x)\delta J(x)\delta J(x)\delta J(x)} = \int d^4x \Delta_F(0)\Delta_F(0) \tag{3.3}$$

(cf. eq. 2.20), which is the so called vacuum-graph. In general, a vacuum graph is a graph with no external legs, a closed graph. This means that there are no particles entering, nor coming out of the interaction, it is produced and annihilated in vacuum (but with the other particles as catalysts).

Solution: In the functional method described in the last chapter, the vacuum diagrams cancel with the ... in equation 3.2. In the usual methods the vacuum graphs have to be dicarded as representing an unmeasurable phase for the S matrix elements.

Discussion: This type of infinity, or unmeasurable phase is fairly well remedied with the functional method, though there are still a few question marks about what it really means to divide an infinite quantity with another.

3.2 Particle annihilating itself, $\Delta_F(0)$

$$\Delta_F(0) \quad \Leftrightarrow \quad (\Box + m^2)\phi_F(0) = \delta(0)$$

This means that the particle annihilates itself, or that a particle and an anti particle are created at the same point in space-time, they travel to another point and then they annihilate. An example of such a graph could be figure 3.2

Note the difference between the vacuum graphs described in the previous section and these graphs. Here we have two external legs while we had none before.

Solution: These infinites are removed with something called normal-ordering. This means, basically, that the annihilation operators are placed to the right of the creation operators.

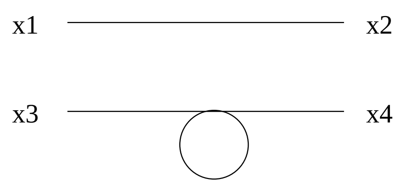

Figure 3.2: Self-annihilating particle.

Discussion: This is rather natural as the creation of a praticle always has to take place before the annihilation. The normal-ordering means that the $\Delta_F(0)$ does not exist in reality (well, as real as the graphs can ever be.)

3.3 Mass-shift, pole in $m^2 - i\Pi$ instead of in m^2

The Feynman diagrams for the $-\lambda\phi 4/4!$ theory are quite simple and will be described briefly. There is only one kind of vertex, and one kind of internal or external line. The rules for constructing Feynman diagrams are:

internal line	$\frac{i}{p^2-m^2+i\epsilon}$
loop integration	$\int \frac{dk}{(2\pi)^4}$
vertex	$-i\lambda$
symmetry factor	S

Through a simple graph we will see how renormalization occurs naturaly, even without the effect of canceling infinities. The simplest graph that can be imagined is a line with only two external legs, i.e. the propagator. This graph has the expression

$$G_1 = \frac{i}{p^2 - m^2 + i\epsilon}. \tag{3.4}$$

To the second order in the expansion the expression is

$$\begin{aligned} G_2 &= G_1 + G_1^2 \int \frac{dk_1 dk_2}{(2\pi)^8} \frac{(-i\lambda^2)}{(k_1^2 - m^2)(k_2^2 - m^2)[(p + k_1 - k_2)^2 - m^2]} && (3.5) \\ &= G_1[1 + G_1\Pi(p)] && (3.6) \end{aligned}$$

where $\Pi(p)$ is the integral over k_1 and k_2, defined by

$$\Pi(p) = \int \frac{dk_1 dk_2}{(2\pi)^8} \frac{(-i\lambda^2)}{(k_1^2 - m^2)(k_2^2 - m^2)[(p + k_1 - k_2)^2 - m^2]}. \tag{3.7}$$

To the n:th order in the expansion the expression is

$$G_n = G_1 \sum_{k=0}^{n} (G_1\Pi(p))^k. \tag{3.8}$$

If we use $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad |a| < 1$ and let $n \to \infty$ we obtain

$$G_\infty = G_1 \frac{1}{1 - G_1 \Pi} = \frac{1}{G^{-1} - \Pi} = \frac{i}{p^2 - m^2 - i\Pi + i\epsilon} \tag{3.9}$$

Solution: We notice that the factor $m^2 + i\Pi(p)$ now plays the role of the mass earlier:

$$\frac{i}{p^2 - m^2 - i\Pi} \quad \Leftrightarrow \quad \left(\Box + (m^2 + i\Pi)\right)\phi = 0 \tag{3.10}$$

and we define this as being the renormalized mass,

$$m_R^2 = m^2 + i\Pi(p). \tag{3.11}$$

The term $i\Pi(p)$ is usually called the mass-shift and denoted δm^2. The mass m_R^2 is what we observe in experiments.

Discussion: We have introduced the renormalized mass as a consequence of the interaction, not as a remedy for infinite quantities. Hence, the renormalization is not only a remedy for infinite amplitudes, but exists without them too.

3.4 Ultraviolet divergence

The ultraviolet divergence can be said to have two sources, that the spacetime dimension is four, or that the momentum- transfer is unlimited, which can be regarded as a consequence of continous space. There are two principal types of ultraviolet divergence in the $\lambda\phi^4/4!$ theory, whereof we will only consider the simpler one. The caculation of the other one is more complicated, and adds nothing new to our discussion. The simpler one can be represented graphically as in figure 3.4

With the Feynman parameters and a change of variables, this can be put

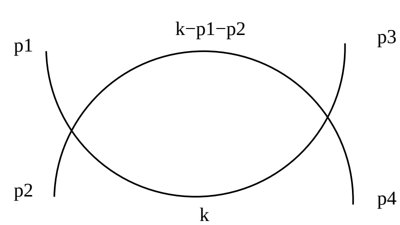

Figure 3.3: The ultraviolet divergence appears in this two vertex graph.

in the form:

$$\int \frac{d^4k}{(k^2 + b^2)^2}. \tag{3.12}$$

With the use of the formula:

$$\int \frac{d^4k}{(k^2+b^2)^n} = i\pi^2 \frac{\Gamma(n-2)}{\Gamma(n)} \frac{1}{(b^2)^{n-2}} \tag{3.13}$$

we see that the integral goes towards infinity due to the pole in in the $\Gamma(m)$ function in $m = 0$.

Solution: In the standard theory, this problem has two solutions. The classical one is to introduce a cut-off energy, Λ and then intergrate to Λ instead of to infinity. Then you add a counter term to the Lagrangian $-A\phi^4/4!$ and define the constant A in terms of Λ so that this will annihilate the Λ-dependent term from the integral. Finally, you let Λ go to infinity which does not affect our solution as this is now Λ-independent. The second method to solve this problem is more elegant at least from a mathematical point of view. This method is called dimensional regularization which means that you put $n = 4 - \epsilon$. You can then separate the part that contains ϵ from the other parts. As before, you add a counter term of the same form $-A\phi^4/4!$ in the Lagrangian but now you let A depend on ϵ so that the final Lagrangian is ϵ-independent.

Discussion: It is very unsatisfactory to be forced to add infinite counter-terms in the Lagrangian to get some sense out of the equation. The only confort is that the counter term can be interpreted physically as representing the fact that what we observe is not the bare particle, but rather the particle with corrections. In the cut-off method the change of variables required to obtain equation 3.13 is not mathematically correct. In the dimensional method we have the strangeness of a non-integer number of dimensions. What does that really mean? There is probably no physical meaning in it, but it is still rather weird. On the other hand, it is not obvious that a discretization of space would solve the problem either. The discretization takes place in normal space whereas the infinity appears in momentum space. Nevertheless, the two of them are so intimately connected that the discretization of normal space should also lead to a maximum possible momentum, some sort of natural (in the sense of explainable) cut-off.

3.5 Infrared divergence

The presence of zero-mass particles[1] in quantum field theory gives rise to a problem known as the infrared divergence problem. It is not considered to be by far as serious as the ultraviolet divergence. The reason for this is that it can be fairly well accounted for in the theory. The infrared divergence is due to the fact that we can always have an infinte number of photons around every electron, thus seemingly giving rise to an infinite cross-section. Hence

[1] Obviously, this excludes the $\lambda\phi^4/4!$ theory, but includes quantum electro dynamics.

problem here is not in the individual matrix elements, as it was for the ultraviolet divergence, but rather in the final cross section.

Solution: However, it turns out that the contribution from external photons will counter the contribution from internal photons. These photons are called soft photons due to their low energy ($\Delta E < 2m_e$ so that no electron-positron pair are contributing). For the external photons, the cross section is given by:

$$\sigma_{ext} = \left(\frac{\Delta E}{\lambda}\right)^P B\sigma \tag{3.14}$$

where $\lambda \rightarrow 0$ is the lower boundary for the photon energy and σ is the cross section without the soft photons and P is a positive constant. On the other hand, when we calculate the cross section for the internal photons we find it to be:

$$\sigma_{int} = \left(\frac{\lambda}{\Lambda}\right)^P \tag{3.15}$$

Hence, the total cross section is given by

$$\sigma_{ext,int} = \left(\frac{\lambda}{\Lambda}\right)^P \left(\frac{\Delta E}{\lambda}\right)^P B\sigma = \left(\frac{\Delta E}{M}\right)^P \bar{\sigma} \tag{3.16}$$

where $\bar{\sigma} = \left(\frac{M}{\Lambda}\right)^P B\sigma$ represents the part of the cross section in which only the ultraviolet infinities remain.

Discussion We see that the internal and the external parts of the cross section cancels thus leaving us with a finite cross section (apart from the ultraviolet divergence). The infrared divergence is not a consequence of the space but rather stemming from the photon being massless. Thus, the discretization of space-time would not affect this divergence.

3.6 Discussion

Of the different kinds of infinities we encounter in quantum field theory some of them can be treated fairly well in the standard model (like 2, 3 and 4), while others still give the right results but look very strange and includes bizzare assumptions (like 1 and 5). Discretizing space could be a solution to the strange infinities, but obviously there are also aother candidates. Superstring theory is one possibility, where the point interactions giving rise to the infinities, are replaced by the ”nicer” string interactions, or membrane interactions. These need not the renormalization for e.g. the ultraviolet divergence. Either discretization, or superstring theory could also be used to effectively combine quantum mechanics with general theory of relativity, as we’re not faced with the problem of renormalization any more. String-theory.

Chapter 4

Discrete Gauge Fields

4.1 Introduction

Many problems in physics today are born from the consequence that we look at space-time continuously; we are going to give in a brief overview an introduction to discrete fields by working with the Maxwell theory.

$$\xi^{\mu\nu\rho\sigma}\partial_V F_{\rho\sigma} = 0 \tag{4.1}$$

$$\partial_V F_{\mu\nu} = J^{\mu} \tag{4.2}$$

Where F is the electromagnetic field and J is the source.

A lot of the calculation and the thinking are going to be left out mainly on the reason that this paper should be short and brief. For further study and questions on the subject I refer to the work of Manoelito M de Souza, Discrete gauge fields.

Let's start by saying that it wouldn't be a problem if everything were to be discrete, because in the bigger scale everything would anyways look continuously. A discrete field gives an illusion of continuity or a macroscopic approximation at a grand scale. Many of today's problems can easily be explain by discrete fields, take as an example wave-particle duality: you have wave-like properties because it's a field and particle-like properties because it's discrete. That's one of the many reasons that discrete fields are interesting and worth take a closer look at.

To start we are going to change only one thing in the standard field theory: instead of a light cone we use a straight line embedded in a 3+1 Minkowski space-time, the light cone generator.

This gives 2 constraints at first look:

1. Constraint between the source's acceleration and the direction of the emitted field.

2. Any field is associated to a continuity equation.

Discrete solution to the standard wave equation gives also following properties (these properties are the result from not using light cone support).

1. Consistence with the wave equation during the field propagation.
2. Every field is finite, propagating and representing only one single point.
3. No singularity in 'infinity'. Singularity is a consequence of the light cone support, a reflex of the light cone vertex
4. There are no advanced discrete field.
5. Is determined only by the state of motion of its source. No gauge freedom.
6. One to one map between the field and its source.
7. Anti-symmetric force field is a consequence of causality and Lorentz covariance.
8. They are transversal fields. No non-physcal degrees of freedom.
9. Well-defined conserved energy and momentum everywhere.
10. The wave equation has only one kind of solution.
11. They produce effect just like the traditional fields.
12. The normal field can be seen as an average effective field from linear combinations of the discrete fields
13. There is a loss of information (the causal constraint) when obtaining the continuous field from the discrete one.

4.2 Gauge fields & Causality

From the Maxwell's equations we have (F is an anti-symmetric tensor):

$$F_{\mu\nu} = \partial_v A_\mu - \partial_\mu A_v \tag{4.3}$$

$$A^\mu - \partial^\mu \partial . A = J^\mu \tag{4.4}$$

$$\Downarrow$$

$$\begin{array}{c} \partial_V F_{\mu\nu} \Rightarrow A + \partial\Lambda \\ F \Rightarrow F \end{array} \tag{4.5}$$

Imposing the gauge condition $\partial.A = 0$, which is an integrability condition for eq.4 we get:

$$A = J \tag{4.6}$$

All thought one can see from eq.1 and eq.2 that gauge freedom and charge conservations are consequences of the anti-symmetry tensor F, with discrete theory one can show that they are not consequence of the anti-symmetry even if you needed to have gauge freedom (last chapter).

To properly define the discrete field in a covariant way causality is used.

$$\tag{4.7}$$

This relation describes the free evolution of an interacting field between two interaction events. Note that hyper-cone is used and not light-cone, the propagation of the physical objects is constraint so each of its moving points is on a world-line tangent to a generator of its hyper-cone. With a little algebra and defining f as a constant four-vector tangent to the hyper-cone,$f^{\mu} = \frac{dx^{\mu}}{d\tau}$, we get:

$$\Delta\tau + f.\Delta x = 0 \tag{4.8}$$

Δx is the separation between the event and the hyper-cone vertex.

Which defines the hyper-plane tangent to the hyper-cone (7).

The tangent of f^{u} is defined as the hyper-cone generator, which one gets from eq.7 & eq.8. With this two equations we see space-time causal structure as a congruence of lines, a set of points.

A lot of information can be taken from eq.7 and eq.8, for example using the following constraint (using eq.8 for a mass-less field):

$$\begin{aligned} f.(x - z(\tau)) = 0 \\ \partial_{\mu} f.(x - z) \mid_{f} = 0 \end{aligned} \tag{4.9}$$

$\Delta x = x - z(\tau)$ is the separation between a source and its field

We get a constraint between the direction f (the direction of the emitted signal) and the change in the charge state of motion at the time.

$$a.f \mid_f = 0 \tag{4.10}$$

Where $a^\mu = \frac{dV^\mu}{d\tau}$ and $V = (\vec{V}, V_4) = \frac{dz}{d\tau}$.

This equation hold for all kinds of fields and sources and it's also the cause behind results like the conservation of laws among others.

4.3 Discrete Fields

Because fields are tied to the proper time of their point source we get with eq.7 and using $\Delta\tau = \Delta t$:

$$\tau = \tau_0 \pm \sqrt{-(\Delta x)^2} \tag{4.11}$$

Making an explicit dependence $\tau(x)$ on A(x) to implement local causality, who itself requires a field support on hyper-cones. We get A_f to be:

$$A_f(x,\tau) = A(x,\tau)\Bigg|_{\substack{\Delta\tau + f.\Delta x = 0 \\ \Delta\tau^2 + \Delta x^2 = 0}} = A(x,\tau)|_f \tag{4.12}$$

Extended causality requires support on a line f

For a mass less field, with eq.6 can be written as

$$\eta^\mu \nu \nabla_\mu \nabla_\nu A_f(x,\tau) = J(x,\tau) \tag{4.13}$$

And solving the resulting equation with the discrete Green's function we get:

$$A_f(x,\tau) = \int d^4y d\tau_y G_f(x - y, \tau_x - tau_y) J(y) \tag{4.14}$$

Solving G_f

$$G_f(x,\tau) = \frac{1}{2}\theta(-bf_4 t)\theta(b\tau)\delta(\tau + f.x) \tag{4.15}$$

As we can see the point signal here propagates on a straight line f.

The reduction of the field support from a light-cone to a light-cone generator reduces the discrete fields to just a point in the phase space.

$$G(x,\tau) = \frac{1}{r}[\delta(r - t) + \delta(r + t)] \tag{4.16}$$

Comparing the discrete Green's function (15) with normal Green's function we can se directly that there exist no singularity in the discrete one $\Rightarrow$ the field propagates without changing the amplitude. A solution of eq.13 with the discrete Green's formula gives with $f^2 = 0$ after some calculations:

$$\begin{gathered}\eta^{\mu\nu}\nabla_\mu\nabla_\nu G_f(x) = \\ -(f_4^2 + |\vec{f}|^2)\delta(\tau)\delta(f_4 t - |\vec{f}|x_L)\delta(f_4 t + |\vec{f}|x_L) \\ = 2f_4^2\delta(\tau)\delta(2f_4 t)\delta(|\vec{f}|x_L) = \delta(\tau)\delta(t)\delta(x_L)\end{gathered} \tag{4.17}$$

Where $f^\mu = (\vec{f}, f^4)$,$\bar{f}^\mu = (-\vec{f}, f^4)$ and f & $\bar{f}$ are opposing generators in the same light-cone.

This equation has 2 solutions b=1 and b=2 corresponding the creation and annihilation of discrete fields.

Equations 15 and 17 give all fields the constraint that they cannot be independent of their sources. The discrete field is not a gauge field because it's determined only by its source. Since it has no gauge freedom it isn't a gauge field.

Now we are going to use the following equation, for the proof of this formula we refer to the original work.

$$G(x,\tau) = \frac{1}{2\pi}\int d^4 f\delta(f^2)G(x,\tau)_f \tag{4.18}$$

From the equation 18 we can see the relation between the discrete and the standard field

$$A(x,\tau) = \frac{1}{2\pi}\int d^4 f\delta(f^2)A(x,\tau)_f \tag{4.19}$$

Giving the conclusion that A is a smearing of A_f over the light-cone and shows an average of the discrete field, it only can give a "true" description at large quantities of for example photons in an electromagnetic wave. On the other hand all the information about f is lost smearing process.

Look at picture; the false picture that A(x) gives is the root of the problem at short distances.

4.4 Further Analysis

To show that gauge freedom and charge conservations are consequences of the extended causality and Lorentz covariance we calculate eq.13 with a mass-less particle, let's say a photon. The mass-less particle ignores eq.3, i.e. the information about the Maxwell tensor structure, in other words ignoring the connection between the gauge symmetry and charge conservation.

We get the following equation:

$$\nabla.A_f = \nabla.J \tag{4.20}$$

We want to show that the Lorentz's condition

$$\nabla . A_f = 0 \tag{4.21}$$

is not a consequence of the continuity equation,

$$\nabla . J = 0 \tag{4.22}$$

and that both are instead consequences of the properties of a discrete world.

With the four vector current given by:

$$J^\mu(y, \tau_y = \tau_z) = eV^\mu(\tau_z)\delta^3(\vec{y} - \vec{z})\delta(t_y - t_z) \tag{4.23}$$

using eq.8 and b=+1 we come after a while to the following expression:

$$A_f(x, \tau_x) = eV^\mu(\tau_z)\theta(t_x - t_z)\theta(\tau_x - \tau_z)|_{\tau_z = \tau_x + f.(x-z)} \quad \text{with } \Delta\tau = 0 \tag{4.24}$$

$$A_f(x, \tau_x = \tau_z) = eV^\mu(\tau_z)\theta(t_x - t_z)\theta(\tau_z)|_{f.(x-z)=0} \tag{4.25}$$

This equation tells us many things. It's both discrete and differentiable, it has both space and time derivatives. This is the reason why the wave-like and particle-like properties in an object are possible.

This is also a universal field, and it is the only solution for a point source.

Further with t=0 and $t > 0$ these equations becomes:

$$A_f = eV|_f \tag{4.26}$$

$$\nabla_\nu A_f^\nu = \nabla_\nu(eV^\mu)|_f = -ef_\nu a^\mu|_f \tag{4.27}$$

The constraint in eq.10 is a consequence of extended causality and it's a constraint between the direction of propagation and change of motion. Equations 21 and 22 are both consequences of 10. We know that there is a causal link, which doesn't depend on the field tensorial or spinorial nature, between the discrete field and it's source that leads to eq.21 and eq.22.

By using a vector field defined by (for a discrete current for a spin-less electron):

$$j_V^\mu = j^\mu|_V = \int d^4y J^\mu(x-y) = eV^\mu(\tau)|_V \quad \text{and compare it with 26} \quad \Downarrow \tag{4.28}$$

Regardless Maxwell's equation, the F we have charge conservation => Charge conservation is a consequence of extended causality and not gauge symmetry.

4.5 Overview

To sum it all up, a finite and consistent field theory requires a light-cone generator as the field support, so is defined by causality using a hyper-cone instead of a light-cone.

From eq.10 we can see that is not a gauge field and doesn't have any "singularities".

The traditional continuous field A is an average of the discrete field, we can see that from eq.19. The null f direction is the one to one link that links the field event to the source event, this link is loss in the integration when we are acquiring A from. From eq.19 we can see that the field A that we get is a gauge field with "singularities". The generic fibre f, that actually is a (1+1) manifold embedded on a (3+1) Minskowski space-time, in the integration is replace with the usual light-cone. The result of this is that the point charge event becomes suddenly linked to an infinity of field events.

Why this idea with discrete fields should be investigated more carefully goes without saying. Just only by studying the universe in this way, it can give us an understanding of how the world around us works.

Chapter 5

Calculations in Discrete Space-time

5.1 Discretizising the Klein-Gordon Equation

The Klein-Gordon-equation can be written in the form

$$(\Box + m^2)\phi = g(\phi), \tag{5.1}$$

where $g(\phi)$ is a source term.

We can express the wave-operator $\Box$ as $\partial_\mu \partial^\mu$, where $\partial_\mu = (t, \mathbf{x})$ and $\partial^\mu = (t, -\mathbf{x})$ and $x = (t, \mathbf{x})$ and $\mathbf{x}$ is the vector (x_1, x_2, x_3). Space can be discretesized in many ways. Here we will only count the interaction from the closest neighbours and from the closest time. There are two ways to do that, to the 'right' and to the 'left'. We can write this mathematically as

$$\Delta^R = \frac{1}{|a|}(\phi(x+a) - \phi(x)) \qquad \Delta^L = \frac{1}{|a|}(\phi(x) - \phi(x-a)). \tag{5.2}$$

where $|a|$ is the space lenght between the points. By letting $\Box = \partial_\mu \partial^\mu = \Delta^R \Delta^L$, the Klein-Gordon equation will be discretesized as

$$(\Delta^R \Delta^L + m^2)\phi = g(\phi). \tag{5.3}$$

Explicitly, with equal spacing a, and time-spacing τ the equation becomes

$$\begin{array}{c} \frac{1}{\tau^2}(\phi(t+\tau) - 2\phi(t) + \phi(t-\tau)) - \\ \sum_{i=1}^{3} \frac{1}{a^2}(\phi(x_i + a) - 2\phi(x_i) + \phi(x_i - a)) \\ +m^2\phi(x) = g(\phi(x)). \end{array} \tag{5.4}$$

Probably, the equation has to be solved numerical, especially when the term g is nontrivial. In the continous case, Feynman graphs gives us the method to calculate the solution by perturbation method. So far we are dealing with uncomfortable infinities, for instance the free propagator. Instead of trying to find new mathematics, we will only derive the propagator in our discrete space-time and try to see if it contains any infinity.

5.2 Free Propagator in Discrete Space-Time

Previously, we stated that the action for a free particle in the Klein-Gordon equation is

$$S = \int d^4x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2). \tag{5.5}$$

Discretizising the space will turn the action into a sum. In our Δ^R and Δ^L operators, we put the scalar a equal in all directions. This gives us the discrete action as

$$S = \sum_n a^4(\Delta^R\Phi\Delta^L\Phi - m^2\Phi^2), \tag{5.6}$$

where now $\Phi = \Phi(n)$. To derive the propagator we wish to operate in k-space. One familiar relation between spatial-temporal space (or n-space) and wavevector space is the four dimensional Fourier transform

$$\Phi(n) = \int \frac{d^4k}{(2\pi)^4} e^{ikna}\Phi(k). \tag{5.7}$$

In the case $\Phi(x+a)$, we have to sum over all directions to get the correct transform. We write that transform in k-space as

$$\sum_{i=1}^{4} \int \frac{d^4k}{(2\pi)^4} e^{ia(kn+k_i)}\Phi(k), \tag{5.8}$$

where the vector k_i denotes the difference of phase in the i:th direction. Figure 5.1 illustrates the case of two dimensions. After an appropriate trans-

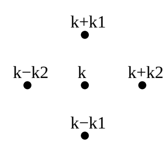

Figure 5.1: The k-vectors around a point on a two-dimensional lattice.

formation and some manipulation, the operator $\Delta^R\Phi\Delta^L\Phi$ becomes

$$\sum_{i=1}^{4} \frac{1}{a^2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} e^{ina(k+k')}\phi(k)\phi(k')(e^{ik_ia} - 1)(1 - e^{-ik'_ia}). \tag{5.9}$$

Expression 5.9, can be regarded as a convolution and we then get

$$\frac{1}{a^2} \int \frac{d^4k}{(2\pi)^4} e^{inak}\phi(k)\phi(-k)(e^{ik_ia} - 1)(1 - e^{ik_ia}). \tag{5.10}$$

The propagator, Δ_F, is defined in chapter 2 by

$$(\partial_\mu^2 + m^2)\Delta_F(x-y) = -\delta^4(x'-x). \tag{5.11}$$

To find the propagator we transform both sides into k-space. After a transformation, the right hand side can be identified as equation 5.10 and the left hand side is unity. Thus

$$-a^2 \int \frac{d^4k}{(2\pi)^4} e^{inak} \Delta_F(k) \sum_{i=1}^{4} (e^{ik_i a} - 1)(1 - e^{ik_i a}) + \int \frac{d^4k}{(2\pi)^4} e^{ikna} \Delta_F(k) m^2 = \int \frac{d^4k}{(2\pi)^4} e^{inak}. \tag{5.12}$$

Then we can identify the propagator as $\phi(k)\phi(-k)$ and extraction gives us the propagator,

$$\Delta_F(k) = \frac{a^2}{m^2 a^2 - \sum_{n=1}^{16} (e^{ika} - 1)^2}, \tag{5.13}$$

which can be written as

$$\Delta_F(k) = \frac{a^2}{m^2 a^2 - \sum_{i=1}^{4} (isin(k_i a) + cos(k_i a) - 1)^2}. \tag{5.14}$$

5.3 Limits and Results

The continuous limit of this discrete propagator is

$$\lim_{a \to 0} \Delta_F(k) = \frac{a^2}{-m^2 a^2 - \sum_{i=1}^{4} (isin(k_i a) + cos(k_i a) - 1)^2} = \frac{1}{k^2 - m^2}. \tag{5.15}$$

In the continuous limit this is in agreement with the propagator in the standard model.

We put $a = 1$, $m = 1$ and consider only one wave vector dimension. In figure 5.2 we plot the discrete propagator expression with the continous one. We can truncate the k-number because of periodicity and only take the intervall $-\pi/a < k < \pi/a$. As we can conclude from the graph, the continous propagator has, as expected, two infinities, namely when $|k| = 1$. Our discrete space does not have any infinities, but the global maxima lie at the same k-numbers and then turn down.

When is the effect of the eventually discretizied world detectable? So far $10^{-15}m$ has been tested, and no sight of a non- continous space. If we put $k = \frac{1}{a}$ into $E = \hbar kc$ we get

$$E = \frac{\hbar c}{a} \approx \frac{10^{-7}}{a} eV, \tag{5.16}$$

which gives an energy of about $0.1GeV$. If we assume a distance between the points to be the Planck length $a = 1.61605 \cdot 10^{-35}m$, the caracteristic wavenumber for eventually effects to be notacible is about 1/Planck lenght, roughly $10^{35}m^{-1}$. This gives av energi of about $10^{28}eV$, which is at the scale of the expected GUT-theories.

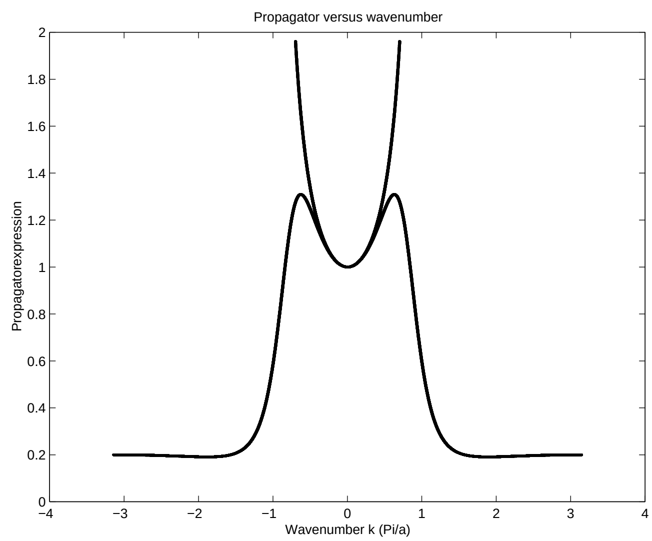

Figure 5.2: Continous propagator (upper curve) and real part of discretesized propagator (lower curve) plotted.

5.4 Propostions and Speculations

So far we have considered the effects of a discrete space-time with one space dimension and onetime dimension. However, this is a very simplified model and in a four dimensional theory the structure of the discretization also has to be acounted for. Our first guess would be that the coordinates are aligned in a cartesian manner because this is the easiest model. On the other hand, the points might as well have other configurations, maybe of unknown form. For example, do they have to be aligned to a specific pattern at all? Maybe they are just defined in conjuction with each other, thus leaving the structure undetermined. This can be argued in analogy with quantum mechanics and also quantum field theory, where the particles and the fields respectively are quantisized and thus undetermined. Maybe the discretization of space-time could be the third quantization? This would mean that the uncertainty of the coordinate-points follows some sort of Heisenberg's uncertainty principle, intriguing, indeed. Maybe the coordinate points have an energy, just as particle has an energy even at rest. It is a well known fact that matter bends space-time, could this be an effect due to coordinates with energy? If the coordinates would have energy, wouldn't this also bring the possibility of interactions between them? We could have coordinate waves, maybe bosons? On the other hand this would cause severe complications for our conception of the world. How would the coordinates propagate? However, if the oscillations were significant we would notice a difference in our experiments, thus it should be possible to determine an upper bound for these oscillations. The coordinate energy might even be the missing energy in the universe, the dark matter. The estimated amount of dark matter might also be used to determine limits on the oscillations.

In order to explain the uncomfortable infinities in the standard model, we can also argue that our physics is so far from the high energy scales where we would see a difference, that the physics is entirely different there. Maybe superstrings, maybe a Great Unified Theory or something completely different. But even then, these are not excluding the possibilty of discrete space-time.

The implications and possibilities in a discrete space-time are enormous. Nevertheless, a deeper study in all these things is very complicated and far beyond the scope of this work. We will have to content ourselves with the possibilities, at least for this time.

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